

[Graph Colouring]

The idea of graph colouring arose when a mathematician and botanist called Francis Guthrie was trying to colour county maps of England and noticed that four colours seemed like it was enough to colour **any** map so that no two counties sharing a border were the same colour. This doesn't include the case of regions only sharing a single point like on a chess board. It's an interesting case of a mathematical problem because although the problem seems simple and the idea seems simple, it wasn't until some 120 years later, after the rise of computers, that a proof was finally developed by Kenneth Appel and Wolfgang Haken – however this caused a little controversy of its own as it required a computer to prove it.

When we convert a map colouring to a graph colouring, we represent each of the regions of the map with a vertex, and connect the vertices by an edge if the regions share a border. It's important to distinguish this: a graph colouring assigns colours to the vertices – Not the faces.

This four-colour theorem also has another important provision, and that's that the graph needs to be planar.

A planar graph is one which can be drawn such that none of the edges cross over one another. Note here that we say "can" be drawn, because remember that the position of arcs and vertices isn't so important. This graph here is not planar, because there's no way to move around the edges or vertices so that the lines don't cross.

On the other hand, even though this graph has intersecting lines, we can use a continuous morphing of the graph, moving vertices and stretching edges, until the lines don't cross. We say that the original graph and the deformed one are isomorphic – because they contain exactly the same graph information in terms of vertices and connecting arcs between them. So the problem of graph colouring is finding the minimum number of colours such that each vertex has a colour assigned, and no adjacent vertices are the same colour. You can imagine cases where clearly a non-planar graph would require more than 4 colours, but otherwise, no matter how many nodes or how many edges and how they're connected, as long as the graph is planar, the maximum number of colours required is 4.

This minimum number of colours is called the chromatic number. In this graph, we can see that it only requires two colours. We could colour it with more, but the minimum number of colours that we can use, and hence the chromatic number, is 2. This graph requires 3. And in fact, as soon as you have a triangle, you will need three. This recognition is handy if we're trying to find the chromatic number – if we can find a subregion that needs 3, we know that it's at least 3, and if we can find 4 vertices all connected to each other, we'll know that it needs at least 4 (and no more if it's planar).