[Euler Paths]

Recall that we were able to say with certainty that there was no Euler path in the Königsburg bridge problem since the existence of three bridges at A, C and D meant that we had a contradiction. As it turns out, the number of bridges – or the degree of the vertices in a graph, is key to being able to quickly determine whether or not there is an Euler path or an Euler Circuit.

Let's consider this reasoning systematically for edges and vertices. Suppose we have a node, A, and there is one bridge, or one edge connecting A to the rest of the graph. Well in this case, if we start at A, then we will have no bridge to return. If we have two bridges, then we can leave and come back. Or if we don't start at A, we'll arrive and then leave.

If we have three bridges, as we did in Königsburg, then we will leave, return but then leave again without being able to get back – so this is the same situation we had with one bridge, and if we have four bridges, we'll be able to leave and come back twice, ultimately being stuck in region A if we started there. It's not hard to see that this pattern will continue. And of course, if we back up a little, if there are NO edges into A, then if we start there, we can't leave, and if we don't start there, we'll never be able to arrive.

So it's clear that if we have vertices where the degree is odd, we can't have more than two of these. The question then, is about whether it is possible if we have zero, one or two. Well – our solution here won't be conclusive, but Euler's theorem states that if there are no odd vertices in a connected graph, then we always find an Euler Cycle, while if there are two, then we can find an Euler path that starts at one and finishes at the other.

Notice that whenever we add edges to a graph, we can only increase the number of odd vertices by two, decrease the number of odd vertices by two, or leave the number of odd and even vertices the same. So we should never see a graph that has 1 or three odd vertices.

After recognizing this, we can intuitively grasp this theorem by remembering the idea of being "trapped". If all of the vertices are of even degree, then we'll never get trapped at a vertex, except the one we started at after we finish the cycle. If there are two vertices of odd degree, we're guaranteed to leave the vertex we started at, and will become trapped at the other, but we should be able to pass along all the arcs of the graph exactly once first.

Although this theorem doesn't tell us what the path actually is, being able to first determine whether a solution exists, then knowing, in the case of an Euler path, where we need to start and finish, actually makes solving these problems by hand surprisingly easy.