[Summing a geometric progression]

As well as the King of Persia, there are a number of fables and anecdotes related to sums of geometric progressions – most of these are based on the ratio being a half.

There's the joke about a guy who walks into a bar and says "I'll have a beer for myself, half a beer for my friend, quarter of a beer for his friend, an eighth of a beer for his friend, a sixteenth.." until the bartender interrupts and says "I understand", then promptly pours two full drinks.

The eye of Horus was said to be fractured into multiple parts worth ½, ¼, 1/8, 1/16.

Arrow's paradox based on this sequence was even the basis of a romantic scene in IQ – a Hollywood film with Tim Robbins and Meg Ryan. Meg Ryan says they can never kiss because to get to Tim, she would have to cover half the distance between them, and then half of that distance, and then half of that distance and then half of that distance, and it'll take an infinitely long time to get to him. Of course, they end up kissing after a few steps.

The algebra of how sums of geometric progressions is a little less romantic, although it does involve a very special trick.

We will have

$a + ar + ar^2 + ar^3 + \ldots + ar^{n-1}$

which we denote by S_n

Now if we multiply this by r, we obtain S_n * r

$ar + ar^2 + ar^3 + ar^4 \ldots + ar^n$

(note that r * r^(n-1) means we will have n-1 lots of r multiplying each other so one more will give us exactly n).

These series are almost identical except that the first includes a, while the second includes ar^n. So if we subtract the first from the second we will have

S_n* r – S_n = ar^n – a

On the left hand side we can factor out S_n to obtain

S_n (r – 1)

While on the right hand side we can factor out a to give

A(r^n – 1)

And so now we can get S_n by itself by dividing by (r-1) and we will have

$S_n = a(r^n - 1)/(r-1)$

You might see a similar trick employed here as was employed for fractions. In fact, the method of obtaining the fractional representation of a decimal can be obtained using precisely this method. For example, the decimal representation of 1/3 is 0.333333… This decimal representation can be expressed as a geometric series.

0.3 + 0.03 + 0.003 + 0.0003 + 0.00003 …

And so on. The first term is 0.3, and each time we are multiplying by a tenth (0.1). The sum of n terms will hence be 0.3(0.1^n – 1)/(0.1-1).

And in this case, we have n approaching infinity, and you might recognize that if we keep on multiplying 0.1 by itself an infinite number of times it will get smaller and smaller and smaller until it reaches 0. At this point, we will have 0.3(0 – 1)/(0.1-1) = 0.3(-1)/-0.9 which simplifies to exactly a third!