[Properties of operations]

In learning how to add and multiply numbers, we often don't stop to think about properties of these operations. This is similar to the idea of grammar in language – we just learn to use it and everything is fine – but when faced with an unfamiliar situation, or when trying to learn a new language, having a firm understanding of grammar can become much more important. The properties of commutativity and associativity with respect to addition and multiplication might seem somewhat trivial, but they're actually what enables us to work algebraically.

Commutativity means that the order either side of the operation doesn't matter. Three groups of five is the same as five groups of three, or five times three dollars is the same as three times five dollars, even though they might refer to different situations.

Associativity is a kind of "bracket anywhere" property and allows us to extend operations like multiplication and addition, which are defined between two numbers, to multiple arguments. That's why we can have the sum of 5 numbers, the product of 5 numbers – but we can't have a subtraction of 4 numbers or a division of 3 numbers – because subtraction and division aren't associative. So with associativity if we want to add 8, 5 and 9, we can either do the 8+5 and then add 9 or we can do the 5+9 and then add the 8 from the front.

Then of course we can combine these properties together, and this is what enables us to do cancelling out when we work algebraically.

For example, if I have 5x+3, I can subtract 3, or maybe it's better to think of this as adding (-3) now grouping these together we get a result of 0 and so we can let it disappear. And even if I had 3 + 5x, I could add -3, and because of associativity, I can assume there's brackets here, then I can swap the order because of commutativity and then I can use associativity again to change around the brackets and re-evaluate.

We also need to remember how distributivity works. So again starting from 5x +3, I can divide by 5, now this is the same as multiplying by a fifth, but I need to remember that the multiplication applies to both terms inside the brackets.

But then if division and subtraction aren't commutative or associative, why does it work when we use multiplication by a fifth? It's similar to the idea of adding negative 3 instead of subtracting 3, but an important thing to remember is that this 1 is neutral in multiplication, but not so in division. So 3 divided by 5 is the same is 3 times a fifth and a fifth times 3, but if we expend out a fifth times 3, it becomes 1 divided by 5 times 3, not 5 divided by 3.

When we learn how to multiply, add, subtract and divide in primary school, we learn that sometimes the order matters and sometimes it doesn't. For instance, we know that multiplying 3 and 5 gives the same result as multiplying 5 and 3, and in fact that this is true when multiplying all *real* numbers, even decimals and fractions. On the other hand, $9 \div 3$ is not the same as $3 \div 9$. We say that the addition and multiplication operations are **commutative**, while division and subtraction are not.

Division and subtraction can both be expressed in terms of multiplication and division, however. Rather than divide by 3, we can multiply by $^1/_3$. Our $9\div3$ then becomes $9 \times {}^1/_3$, which will give the same result as ${}^1/_3 \times 9$, but is still quite different to $3\div9$. Similarly, instead of performing the subtraction $5 – 2$, we can think of this as $5 + (-2)$ which is the same as $(-2) + 5$. For the moment, then, we will just concentrate on the properties that apply for addition and multiplication.

Another important property is *associativity*. Multiplication is actually a type of 'binary' operation, meaning that it takes 2 inputs and gives 1 output. It is defined as an operation between 2 things. In order to combine 3 or 4 or 5 things with multiplication, we need to make sure that we can extend it consistently. So the property of associativity tells us that, given three numbers, e.g. 8, 5 and 9, performing the operation between the result of the first two with the third is the same as performing the operation between the first and the last two. So for addition and these three numbers, it needs to be true that

$$(8+5)+9 = 8+(5+9)\,.$$

You can check to see that this is true, and it will be true for all multiplication and addition of real numbers, however such a property does not necessarily hold for all operations. Using the example of subtraction, it is easy to check that it is not true that for these three numbers, (8 – 5) – 9 is the same as 8 – (5 – 9). So we can say that **subtraction and division are not associative or commutative**.

We tend to take these things for granted, however they are very important when working algebraically. Even in our example with speed, in order to rearrange the equation, we relied on both commutativity and associativity. We can think of the rearrangement as involving the following steps or checks.

$$(100\times t)\times\frac{1}{100} = 100\times\left(t\times\frac{1}{100}\right) = 100\times\left(\frac{1}{100}\times t\right) = \left(100\times\frac{1}{100}\right)\times t = 1\times t\,,$$

Associativity to change the brackets

Commutativity to change the order

Associativity to change the brackets and perform this operation first